The freeness problem on matrix groups

Summer research 2021 (Supervisor: Emmanuel Breuillard)

Georgi Kocharyan

University of Cambridge

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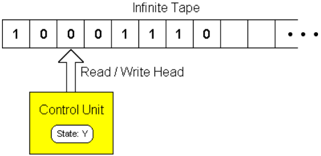
Turing 1936: Turing machine (TM) as model of computation, consisting of:

- An (infinite) tape of 0s and 1s, and a tape head
- A finite amount of 'states' dictating the behaviour of the TM depending on if head is on a 0 or a 1

Turing machines



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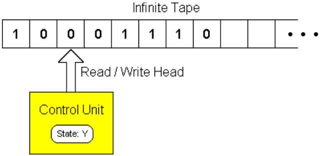

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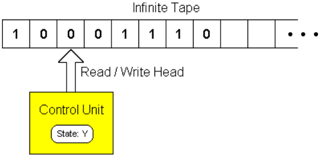

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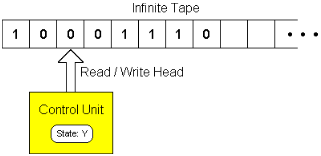

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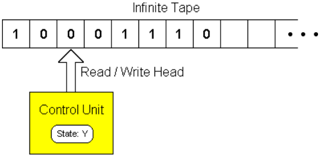

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A problem is *undecidable* if there exists no Turing machine always outputting the correct solution to it.

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One of the first natural examples for an undecidable problem is the *Post correspondence problem* (PCP).
Fix an alphabet $\Sigma = \{a, b\}$. We are given a finite set of n dominoes, each bearing a word in Σ on each of its sides.
Question. Can we, given an unlimited supply of each domino, line a non-zero amount of them up so that the top and bottom spell out the same word?

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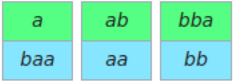

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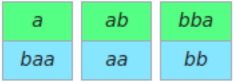

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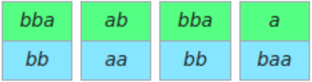

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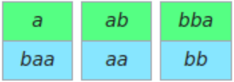

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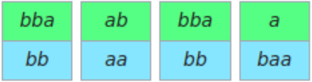

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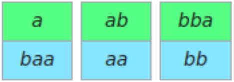

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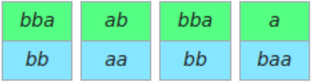

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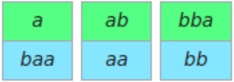

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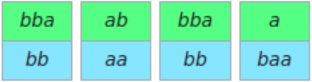

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What if we fix n? In this case the PCP becomes decidable for $n \leq 2$, but remains undecidable for $n \geq 5$.

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Definition (Code)

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- $\mathbb{F} \times \mathbb{F}$: decidable (exercise)

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- For matrix <u>groups</u>: open problem

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Definition (Conjugacy problem)

Given two words in the generators determines whether they are conjugate *within* G or not.

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If G has undecidable word problem, then $w = 1$ in G is equivalent to $(w, 1) \in M(G)$. Then just embed $M(G)$ into $\mathrm{GL}_4(\mathbb{Z})$. $\square$

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Proof.

Add x to the generators and relators of G. Then $w = 1$ in G if and only if (x, x) is conjugate to $(x, w^{-1}xw)$ in $M(G)$. (Exercise!) □

Hyperplane problem

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Lemma

The solvability of the mortality problem for $\mathrm{GL}_3(\mathbb{Z})$ implies the solvability of the upper left corner problem for $\mathrm{GL}_3(\mathbb{Z})$.

Theorem (Halava and Harju, 2001)

The upper-left-corner problem is undecidable for sets of 5 $3 \times 3-$integer matrices.

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Theorem (Result from placement)

The upper-left-corner problem for groups is undecidable for sets of 17 $6 \times 6-$rational matrices.

This is achieved by reducing to the membership problem!

Free subgroups of SO_3

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$$r_x^{\pm\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \mp\sin\alpha \\ 0 & \pm\sin\alpha & \cos\alpha \end{pmatrix} \qquad r_z^{\pm\alpha} = \begin{pmatrix} \cos\alpha & \mp\sin\alpha & 0 \\ \pm\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the rotations by an angle α around the x and z axis respectively. We ask for what α we have that $\langle r_x^\alpha, r_z^\alpha \rangle$ is a free group of rank 2.

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Theorem (Result from placement)

If $\cos\alpha = \frac{2q}{q^2+1}$ *for* $q \in \mathbb{Q}, q \neq 0, 1$ *then the rotation matrices by* α *around the* x *and* z *axes generate a free group.*

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If $\cos\alpha$ is rational but not dyadic (i.e. $\neq \frac{m}{2^k}$), then the rotation matrices by α around the x and z axes generate a free group.

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A few results on this:

Theorem (Result from placement)

If $\cos\alpha = \frac{2q}{q^2+1}$ for $q \in \mathbb{Q}, q \neq 0, 1$ then the rotation matrices by α around the x and z axes generate a free group.

Theorem (Result from placement)

If $\cos\alpha$ is rational but not dyadic (i.e. $\neq \frac{m}{2^k}$), then the rotation matrices by α around the x and z axes generate a free group.

Theorem (Swierczkowski, 1994)

If $\cos\alpha \in \mathbb{Q}\backslash\{0, \pm\frac{1}{2}, \pm1\}$, then the rotation matrices by α around the x and z axes generate a free group.