Axiomatic set theory, Platonism and the Incompleteness Theorems Georgi Kocharyan

Mathematics and philosophy have always seemed to have a special relationship. Many mathematical greats such as Leibniz, Pascal, Gödel or most ancient Greek mathematicians are also well-known for their contributions to philosophy. We take this as a given, seeing both disciplines as a way of finding non-emprical, a priori knowledge by only using logic in a rigourous fashion.

Is their interweaved state of being in any way reasonable? Aren't mathematics and philosophy, albeit related, built up based on fundamentally different viewpoints of the world? Indeed, one could argue that the core task of mathematics is to study logical implication, that is, statements of the form A implies B. Within this viewpoint, a mathematician does not question any deeper truths. She simply assumes certain statements and studies their consequences, turning a blind eye to their foundedness. In contrast, while a philosopher may also study implications, her true objective would be the analysis of the justification of statements A and B. In short: A mathematician does not care about ontological commitment.

While this may superficially seem to be the case, a glance into the complex history of modern mathematical formalism and the gradual foundation of axiomatic set theory will show us that this assumption is flawed. As soon one begins to question the mathematics one is doing and the implicit assumptions one makes, one will without fail have to involve philosophical thought processes to progress.

Before delving deeper, we need to think about *rigour*. We already mentioned rigour as being one of the common characteristics of mathematics and philosophy – in fact, we will see that the drive for rigour occuring in the mathematical world around the beginning of the 19^{th} century was central in the development of the ideas connecting philosophy and mathematics.

So – why does an argument need to be rigourous? Can't we rely on our intuitions to guide us? The clear answer here is no. Not only can intuitions differ between people, but they may even be misleading, as seen many, many times over the course of the history of mathematics. For example, it was thought and widely accepted that every continuous function is differentiable for centuries – and frankly, this does intuitively make sense – until Weierstrass exhibited the first counterexample in 1872, not without considerable backlash from the matahematical community. Shortly, we have no guarantee of a statement's truth until we can rigourously demonstrate it to be so.

But we soon see the problem with this. If we want to prove a statement, we will then need an already established statement or statements that it follows from. But these will also need justification, and so on. This means that if we do not have a certain limited set of statements that everyone simply accepts to be true, doing mathematics will prove impossible in the long run.

This is where set theory comes in. We realise that we don't know anything at all in a fundamental manner about numbers, functions, triangles or any other object we care about. But we more or less have an idea of how to work with sets – we have the intuition that sets can contain other objects, and that is their entire defining property. So is it possible to define (almost) every single mathematical object we care about as a set? It turns out it is. It took almost a century and many failed attempts until Ernst Zermelo presented his set of axioms for set theory in 1910, which after improvements by Abraham Fraenkel is still used as the

basis for modern mathematics today (known as ZF, the Zermelo-Frankel Axioms). They contain statements such as “the empty set is a set” or “the union of sets is a set” as well as some more unintuitive ones. An important stepping stone in this development was the work of Gottlob Frege, who had offered an axiomatisation in 1893 which was shown to be faulty in Bertrand Russell by the famous *Russell’s Paradox*.
But what does it mean for an axiomatisation of set theory to be faulty? Why can’t we simply define a set as any collection of objects, which are by necessity other sets, since these are the only objects we know about? This is sadly too good to be true – what Frege had used was a rule he dubbed the *comprehension principle,* which is exactly this – it simply states that for any property P, there exists the set of all sets that have property P. Russell discovered that accepting this as an axiom inevitably leads to contradiction – this was his argument: Take, as P, the property “does not contain itself.” Frege’s axiom then guarantees that there exists the set of all sets that do not contain itself. Now, asks Russell, does this set contain itself? If it did, then it wouldn’t. If it didn’t, then it would. A contradiction.

A quick digression to be clear on things – why is it bad to have contradictions in mathematics? Can’t we just accept the axioms we like and just ignore their contradictory conclusions while keeping the ones we like? The principle from classical logic *ex falso quodlibet* gives us a definitive no – this states that once given a contradiction, we can derive any statement we can think of, thus rendering any and all conclusions pointless. To see why this is true, here an example, using the principle of disjunctive syllogisms (if we know that the statement “P or Q” is correct, but P is false, then we can conclude that Q is true):
Imagine a contradiction, such as “It is raining and it is not raining.” We can conclude that the statement “It is raining, or London is in France” is true. But now the disjunctive syllogism applied to this statement plus the statement “it is not raining” gives the absurd statement “London is in France”.

So now that we know that we cannot simply let everything be a set – what is a set then? At this point, the connection to philosophy becomes a lot clearer. What we require from our axioms is more or less the following bare minimum:
First, they should not lead to contradictions. Second, they should be rich enough to allow us to do interesting mathematics, as well as not invalidate all of the results discovered in the thousands of years of mathematics up to now. And finally, the most philosophically flavoured of them all, they should be *motivated*, that is, we should accept them as a good, intuitive model of reality.

At this point we are ready to introduce the main dichotomy of standpoints on how the axioms mathematics is built upon should be formulated. *Set-theoretic platonism*, with clear connections to regular platonism, maintains that there is but one correct answer to the question, that is, there exists a single universe of sets we should try to describe as accurately as possible. At the opposite end of the spectrum, we have *formalism*, maintaining that mathematics has no claim to anything of the sort and is simply the study of manipulation of strings under certain rules, and even these rules may be changed as we will. For a formalist, the only interest would be changing the axioms and recording what follows, while a platonist searches for the one true all-encompassing axiom system.

An example illustrating the different approaches would be the *Continuum Hypothesis* (CH), a question posed by Georg Cantor, the father of set theory, in 1878. What it states is not

necessarily important for this article – just for the sake of clarity, CH asks the question whether or not there exists a set of a given size inside the framework of the Zermelo-Frankel axioms, or better said, asserts that it does not. Cantor fervently believed CH was true, that is, that there did not exist such a set, and tried to prove it for many years without success. We would expect the final answer to be yes or no, but it wasn't resolved definitely until 1963 when Paul Cohen demonstrated that *neither* is true – it is simply an independent statement of ZF! This means that there exist certain interpretations of ZF in which CH is true and certain interpretations where it isn't – in short, ZF does not "pin down" the universe of sets in a unique way.
An important player in all of this is Kurt Gödel, who seen as the pioneer of modern logics. In 1940, he was able to prove that CH cannot be disproved from ZF, thus laying the groundwork for its completion in 1963, where it was further proved that it cannot be proved from ZF, thus establishing its independence.

So we already know that ZF does not "pin down" the universe of sets, that is, it leaves room for ambiguity by there being certain statements (e.g. the continuum hypothesis) that can either be true or not true. In fact, as Gödel was able to prove in 1931, for any axiomatic system able to model basic arithmetic, there will *necessarily* exist statements it cannot prove, which has gone down in history as Gödel's first incompleteness theorem. In other words, we can forget about ever having the true axiomatisation of reality. So, the second of our requisites for a good axiomatic system (us able to do all/a lot of mathematics inside) more or less fails, for some definition of failure.
But, we may ask, surely ZF doesn't allow contradictions? Again, Gödel will have to disappoint us. His second incompleteness theorem proves that it is *impossible* to prove the consistency (that is, the non-contradictory nature) of an axiom system from within itself. We would have to assume more axioms, but then we could never guarantee these to be consistent, and so on. This simply means that, unless someone directly exhibits one, we do not know whether ZF allows contradictions, and we will never know.

Gödel was a staunch platonist, and he believed CH to be false. So how does this fit together with his proof that it cannot be disproved from ZF? Very easily – he simply believed ZF to be incomplete, to not be the one true axiom system. And this is closely related to how modern mathematics approaches the subject. What many mathematicians are currently working on is not trying to prove CH, since we already know that to be impossible, but to think of *motivated* and *intuitive* axioms one could add to ZF so that either CH or its negation would follow – but what is this, if not philosophy?

Bibliography:
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